

Control Contraction Metrics: Differential L^2 Gain and Observer Duality

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Abstract—This paper studies the use of control contraction metrics in the solution of several problems in nonlinear control. We discuss integrability conditions of controls and give concrete results for mechanical systems. We also study the relationship between existence of a metric and differential L^2 gain, a form of differential dissipativity, and the use of convex optimization for robust stabilization of nonlinear systems. Finally, we discuss a “duality” result between nonlinear stabilization problems and observer construction, in the process giving a novel construction of a nonlinear observer.

I. INTRODUCTION

Constructive control design for nonlinear systems remains a challenging problem, even in the case of full state feedback [1], [2], [3].

The classical Lyapunov stability theory leads to necessary and sufficient conditions in terms of existence of control Lyapunov functions, however these may be difficult to find [5]. Constructive methods, such as feedback linearization [2], backstepping [6], and energy-based methods [7] are generally applicable only to a limited class of systems. Nonlinear MPC is emerging as a feasible tool (see, e.g. [8]) but despite some clear benefits, it generally remains difficult to predict or analyse performance of nonlinear MPC schemes by any method other than exhaustive simulations.

Recently, there has been increased interest in methods for systems analysis and control design based on convex optimization, for example linear matrix inequalities, integral quadratic constraints (IQC), and sum-of-squares programming [9], [10], [11]. For nonlinear control design, the density functions of [5], [12] and related techniques of occupation measures [13] and control Lyapunov measures [14] explicitly address convexity of criteria. Another approach is to piece together stabilized trajectories, with regions of stability verified via sum-of-squares programming [15].

Several methods have been proposed for robust control design for nonlinear systems, in particular by extending H^∞ -type results for linear systems. We recall in particular Hamilton-Jacobi-Bellman approaches in [16], and the mixed LMI and PDE approach of [17]. However the computational demands of these approaches remain daunting for all but the simplest systems.

Constructing a control-Lyapunov or density function requires prior knowledge of the solution to be stabilized. However, in many applications with complex system architectures, the role of feedback control is to track a setpoint

or trajectory that changes in real-time. For nonlinear systems the problem of stabilizing changing trajectories can be quite different to stabilizing a single *a priori* known trajectory.

Contraction analysis [18], [19], is based on the study of *differential* dynamics. Roughly speaking, if *all* solutions of a nonlinear system are locally stable, then all solutions converge. Thus global stability results are derived from local criteria, and the problem of motion stability is decoupled from the choice of a particular solution. The search for a contraction metric can be formulated as a convex optimisation problem using sum-of-squares programming [20] and can be extended to the study of limit cycles [21].

Historically, basic convergence results on contracting systems can be traced back to the results of [22] in terms of Finsler metrics, and results of [23] and [24]. Extensions to analysis of limit cycles have recently been developed by the authors [21] and connections with Finsler structures and Lyapunov theory have recently been explored in detail by [25].

A contraction metric can be thought of as a Riemannian metric with the additional property that differential displacements get smaller (with respect to the metric) under the flow of the system. A control contraction metric has the property that differential displacements can be *made* to get shorter by control action. This is analogous to the relationship between a Lyapunov function and a control Lyapunov function.

The conditions derived in [4] for stabilizability take the form of state-dependent linear matrix inequalities. This is an attractive property because it opens the door to solution via convex optimization methods, such as sum-of-squares [11]. In this paper examine the question of integrability of controls, discussing in detail the example of a mechanical system. We then derive pointwise LMI conditions for differential and incremental L^2 gain bounds on nonlinear systems, and give pointwise LMI conditions for a class of nonlinear robust stabilization problems. Finally, we discuss a form of “duality” relationship between a control contraction metric and a method of observer design.

We make extensive use of the concepts of *differential dissipativity* and *differential IQCs*. To the authors knowledge, this family of methods was first used to quantify input-output behaviour of nonlinear systems for system identification [26], [27]. The general case was given in [21], and further studied in [28] and [29].

II. PRELIMINARIES

For most of this paper, we will consider a nonlinear time-dependant control-affine system

$$\dot{x}(t) = f(x(t), t) + B(t)u(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ are state and control, respectively, at time $t \in \mathbb{R}^+ := [0, \infty)$. The function $f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is assumed to be smooth, and $B : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m}$ is a time-dependent matrix.

Contraction analysis is the study of (1) by way of the associated system of differential dynamics:

$$\dot{\delta}_x(t) = A(x, t)\delta_x(t) + B(t)\delta_u(t) \quad (2)$$

where $A(x, t) = \frac{\partial}{\partial x}f(x, t)$ is the Jacobian matrix.

In [4] we also considered more general systems $\dot{x} = f(x, u, t)$ but for the moment we note that many systems not naturally appearing in the form (1) can be put in that form, either exactly or approximately, by change of variables or introducing new states.

A solution of (1) is a pair of vector signals $(x(t), u(t))$ satisfying (1) over the interval \mathbb{R}^+ . For simplicity, in this paper we will assume that (1) is such that solutions exist and are unique. As such, we will also use the notation $\phi(t, x_0, u)$ to denote the solution of x at time $t > 0$ of (1) starting from initial condition $x(0) = x_0$ and under the application of the control input $u(\tau), \tau \in [0, t]$.

A static state-feedback controller is a function $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The main objective is to design such a function so that the behaviour of the closed-loop system

$$\dot{x}(t) = f(x(t), t) + B(t)k(x(t), t) \quad (3)$$

is in some sense desirable, e.g. globally stable or optimal in some sense. Solutions of the closed-loop system with a given state-feedback controller k will be denoted $\phi_k(t, x_0)$.

As is standard, a solution (x^*, u^*) defined on $[0, \infty)$ is said to be globally asymptotically stabilized by a feedback controller $u = k(x, t)$ if both of the following hold:

- 1) For any α there exists an ϵ such that $|x_0 - x^*(0)| < \epsilon$ implies $|\phi_k(t, x_0) - x^*(t)| < \alpha$,
- 2) For any initial condition $x_0 \in \mathbb{R}^n$, the closed loop solution satisfies $|\phi_k(t, x_0) - x^*(t)| \rightarrow 0$.

Global exponential stabilization refers to the stronger condition that there exists a K and λ such that

$$|\phi_k(t, x_0) - x^*(t)| \leq Ke^{-\lambda t}|x_0 - x^*(0)|$$

for all $x(0)$.

In this paper, we will study the following property:

Definition 1: A system of the form (1) is said to be *universally stabilizable by state feedback* if for any solution (x^*, u^*) defined on $t \in [0, \infty)$ there exists a state feedback controller $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that (x^*, u^*) is globally stabilized by $u = k(x, t)$.

Analogously, we also consider the notion of *universally exponentially stabilizable with rate λ* .

Note that universal stabilizability is a stronger condition than global stabilizability of a particular solution.

Several times in the paper we will make use of the following simple result:

Lemma 1: Given a smooth function $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$, and a smooth path $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ connecting any to points $x_0, x_1 \in \mathbb{R}^n$, i.e. $\gamma(0) = x_0$ and $\gamma(1) = x_1$, the following inequality holds:

$$|g(x_1) - g(x_0)|^2 \leq \int_0^1 \left| G(\gamma(s)) \frac{\partial \gamma}{\partial s}(s) \right|^2 ds$$

where $G(x) = \frac{\partial g(x)}{\partial x}$

Proof: To simplify notation, let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be the inner product and norm on $L^2[0, 1]$, respectively, and let $z : [0, 1] \rightarrow \mathbb{R}^p$ be defined as

$$z(s) = G(\gamma(s)) \frac{\partial \gamma}{\partial s}(s),$$

so the right-hand-side of the inequality in the theorem is $\|z\|^2$.

Now, define $\mathbf{1} : [0, 1] \rightarrow \mathbb{R}$ to be the function which is equal to one for all $s \in [0, 1]$. So, by construction, for each vector element $j = 1, 2, \dots, p$ of g and z we have

$$g_j(x_1) - g_j(x_0) = \int_0^1 z_j(s) ds = \langle \mathbf{1}, z_j \rangle \quad (4)$$

$$\leq \|\mathbf{1}\| \|z_j\| = \|z_j\|, \quad (5)$$

from the Cauchy Schwarz inequality. The result is obtained by squaring each side of the above inequality, and summing over the coordinates j . ■

III. CONTROL CONTRACTION METRICS

We now give the basic idea of a control contraction metric (CCM), proposed by the authors in [4]. Suppose a system has the property that every solution is locally stabilizable, i.e. the time-varying linear system (18) is stabilisable, where the partial derivatives are evaluated along any particular solution $x(t), u(t)$ of (1). Each local stabilisation may have small region of stability, but if a “chain” of states joining the current state x to $x^*(t)$ is stabilised, in the sense that if each “link” in the chain gets shorter, then $x(t)$ is driven towards $x^*(t)$.

Construction of a CCM is based on taking this concept to the limit as the number of links in the chain goes to infinity, and becomes a smooth path $\gamma(s)$ connecting $x^*(t)$ and $x(t)$ in the state space. The differential dynamics (18) describe the dynamics of infinitesimal path segments. Now, suppose one can find a Riemannian metric $V(x, \delta) = \delta' M(x, t) \delta$ which verifies that a differential feedback law $\delta_u = K(x, t) \delta_x$ is stabilising, i.e.

$$(A + BK)'M + M(A + BK) + \dot{M} < 0 \quad (6)$$

then we refer to $M(x)$ as a control contraction metric. The right hand side above can be replaced with $-\lambda M$ for exponential stability, or various other supply rates for other design criteria. The control signal applied is then computed

by integrating the differential control signals δ_u along the path γ , i.e.

$$u(t) = u^*(t) + \int_{\gamma} K(\gamma(s)) \frac{\partial \gamma}{\partial s} ds. \quad (7)$$

Roughly speaking, one stabilises a smooth nonlinear system by stabilising an *one-parameter family of linear systems*.

In a sense, this is a generalisation of the concept of a control Lyapunov function to differential dynamics. The advantage is that the rich repertoire of design techniques for linear systems using linear matrix inequalities (LMIs) can be adapted to nonlinear systems as *pointwise*-LMIs. Furthermore, pointwise LMIs are now computationally tractable for many important systems thanks to recent advances in semialgebraic optimisation [30], [31].

The following definition is central to this paper:

Definition 2: A function $V(x, \delta_x, t) = \delta'_x M(x, t) \delta_x$, with $c_1 I \leq M(x, t) \leq c_2 I$ for some $c_2 \geq c_1 > 0$, is said to be a *control contraction metric* for the system (1) if $\frac{\partial V}{\partial x} B(t) = 0$ and

$$\frac{\partial V}{\partial \delta_x} B(t) = 0 \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) + \frac{\partial V}{\partial \delta_x} A(x, t) \delta_x < 0 \quad (8)$$

for all x, t .

We will also make use of a Riemannian distance function between any two points at a given time $d(x_1, x_2, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined like so: let $\Gamma(x_1, x_2)$ denote the set of all smooth paths connecting x_1 and x_2 , where each $\gamma \in \Gamma(x_1, x_2)$ is parametrised by $s \in [0, 1]$, i.e. $\gamma(s) : [0, 1] \rightarrow \mathbb{R}^n$. The path length of γ is then defined as

$$L(\gamma, t) := \int_0^1 D\left(\gamma(s), \frac{\partial}{\partial s} \gamma(s), t\right) ds$$

where $D(x, \delta_x, t) = \sqrt{\delta'_x M(x, t) \delta_x}$. The distance between two points is then defined as

$$d(x_1, x_2, t) = \min_{\gamma \in \Gamma(x_1, x_2)} L(\gamma, t)$$

The existence of a minimizing path, which we denote $\gamma_{x_1}^{x_2}(t, s)$, is implied by the Hopf-Rinow Theorem [32].

We will assume without loss of generality that $\gamma(s)$ has “unit speed”, i.e. $|\frac{\partial \gamma}{\partial s}| = 1$ and therefore the minimum length path is the same as the minimum energy path, with $D(x, \delta_x, t) = \delta'_x M(x, t) \delta_x$ [32]. Furthermore, clearly $\delta_x(t)' M(x, t) \delta_x(t) \rightarrow 0$ if and only if $\sqrt{\delta'_x M(x, t) \delta_x(t)} \rightarrow 0$, so for convergence analysis we can study these to quantities interchangeably.

We now briefly summarize some of the results of [4] that we will use in this paper.

Theorem 1: [4] If a control contraction metric exists for a system of the form (1), then the system is universally stabilizable by static state feedback.

Theorem 2: [4] Consider the system (1) with differential dynamics (18). If there exists a matrix function $W(x, t)$, a function $\rho(x, t) \geq 0$, and constants $\alpha_2 \geq \alpha_1 > 0$ such that

$$\alpha_1 I \leq W \leq \alpha_2 I, \quad (9)$$

$$-\dot{W} + WA' + AW - \rho BB' < 0, \quad (10)$$

for all x, t , then the system (1) is universally stabilizable by static state feedback, and $\delta'_x W(x, t)^{-1} \delta_x$ is a control contraction metric.

In the above condition, $\dot{W}(x, t)$ is a matrix with the i, j element given by $\frac{\partial W_{i,j}}{\partial t} + \frac{\partial W_{i,j}}{\partial x} (f(x, t) + B(t)u)$.

It is worth remarking that Theorem 2 extends a necessary and sufficient condition for stabilizability of linear systems [33, p. 126] to a sufficient condition for stabilizability of nonlinear systems. A stronger result is the following:

Theorem 3: [4] Consider the system (1) with differential dynamics (18). If there exists a matrix function $W(x, t) \in S_+^n$, $\rho(x, t) \geq 0$ and $\alpha_2 \geq \alpha_1 > 0$ such that

$$\alpha_1 I \leq W \leq \alpha_2 I, \quad (11)$$

$$-\dot{W} + WA' + AW - \rho BB' \leq -2\lambda W(x), \quad (12)$$

for all x, u, t , then the system is universally exponentially stabilizable with rate λ by state feedback.

A. Integrability Conditions in Nonlinear Control Design

There are many approaches to nonlinear control design that are based on extending the attractive computational properties of linear systems to nonlinear problems. Thinking of a linear system as the “differential” of a nonlinear system, it is therefore not surprising that going back from a linear design to a nonlinear design requires some sort of “integration”. This is frequently the key difficulty, and in this section we discuss briefly how it manifests itself in different cases.

In feedback linearization [2], it is relatively easy to give a description of the differentials of a change of coordinates that would linearize the system, but it is not nearly so straightforward to actually construct the change of coordinates. The vector field of differential coordinates must be integrable, and while necessary and sufficient conditions can be derived from Frobenius theorem, actually constructing such a change of coordinates requires solve a partial differential equation, and so can be difficult even when one can be shown to exist.

In [17], a design procedure was given for a non-unique system representation $\dot{x} = A(x)x + B(x)u$ based on LMI design for linear systems. The procedure required solving a set of pointwise LMIs, quite similar to the ones for a control contraction metric, but with the additional requirement that the resulting matrix variable $X(x)$ should satisfy the equation $\frac{\partial V}{\partial x}(x) = 2x'X^{-1}(x)$ for some function $V(x)$. This is obviously a highly nonlinear constraint on the elements of $X(x)$.

In this paper, we consider two possible scenarios. If a CCM exists such that the resulting differential gain $\delta_u(t) = K(x, t)\delta_x(t)$ is integrable, i.e. there exists a function $k(x, x^*, t)$ such that $K(x) = \frac{\partial k}{\partial x}(x)$, then the feedback control $u = k(x, x^*, t)$ renders the system contracting and therefore asymptotically stable.

On the other hand, even if $K(x)$ is not integrable over \mathbb{R}^n , we can still compute the *path integral* of $K(\gamma) \frac{\partial \gamma}{\partial s}$ along some path $\gamma(s)$. Note that if $K(x)$ is integrable, then the value of such a path integral is path independent, and the control input can be thought of as a “virtual force field”. When $K(x)$ is not integrable the path matters, but we have

shown that if the path is a geodesic with respect to the control contraction metric, then the closed-loop system is guaranteed to be *stable*, but not necessarily contracting.

B. Example: Mechanical Systems

To illustrate the question of integrability of controls, we examine in some detail the case of a controlled mechanical systems. Here we make use of variation of the analysis in [34] for uncontrolled mechanical systems.

The dynamics of a multilink robotic manipulator with generalised coordinates q , kinetic energy $\dot{q}'H(q)\dot{q}$, potential energy $V(q)$, and generalised forces/torques u are derived from the Euler Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u$$

where $L = \dot{q}'H(q)\dot{q} - V(q)$ is the Lagrangian.

It is well-known [1] that these equations can be written in the form

$$H(q)\ddot{q} + (C(q, \dot{q}) + K_D(q, \dot{q}))\dot{q} + G(q) = u \quad (13)$$

where $H(q) > 0$ for all q and $\dot{H} - 2C$ is skew-symmetric.

For the results in this section, K_D may be a natural damping term, or an artificial damping term introduced by a “derivative” term in the control signal.

Theorem 4: For the class of systems (13) the metric

$$\frac{1}{2} \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix}' M(q, \dot{q}) \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix} \quad (14)$$

where

$$M = \begin{bmatrix} HK_P H & HK_P (C + K_D) \\ (C + K_D)' K_P H & H + (C + K_D)' K_P (C + K_D) \end{bmatrix}$$

is a control contraction metric, which can be made contracting using a type of gravity-cancelling PD control.

Remark 1: If $K_P = 0$ then the control contraction metric given above reduces to $\frac{1}{2} \delta_q' H(q) \delta_q$, i.e. the Riemannian metric associated with kinetic energy.

Proof: Since $H(q)$ is positive-definite, we can consider a factorization $H(q) = T(q)'T(q)$ with $T(q)$ non-singular for every q . Define $x = [\dot{q}'q']'$.

Select some nonsingular $n \times n$ matrix R , and consider a differential change of coordinates

$$\delta_z = \begin{bmatrix} RH & R(C + K_D) \\ 0 & T \end{bmatrix} \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix}$$

Note that the metric (14) is equal to $\frac{1}{2} \delta_z' \delta_z$ by construction. Now, the above choice of differential coordinates implies that

$$\dot{z} = \begin{bmatrix} RH & R(C + K_D) \\ 0 & T \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Rv \\ T\dot{q} \end{bmatrix}$$

where we have defined the virtual control input $v = u - G(q)$, which cancels the effect of the potential energy field. The differential dynamics are

$$\dot{\delta}_z = \begin{bmatrix} 0 & 0 \\ T & \dot{T} \end{bmatrix} \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix} + \begin{bmatrix} R\delta_v \\ 0 \end{bmatrix}$$

Now, the rate of change of differential length is $\frac{d}{dt} (\frac{1}{2} \delta_z' \delta_z) = \delta_z' \dot{\delta}_z$. This evaluates as

$$\begin{aligned} & \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix}' \begin{bmatrix} HR' & 0 \\ (C + K_D)' R' & T' \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ T & \dot{T} \end{bmatrix} \begin{bmatrix} \delta \dot{q} \\ \delta q \end{bmatrix} + \begin{bmatrix} R\delta_v \\ 0 \end{bmatrix} \right) \\ &= [\delta_q' H \delta_q + \delta_q' T' \dot{T} \delta_q + \delta_q' H R' R \delta_v + \delta_q' (C + K_D) R' R \delta_v] \end{aligned}$$

where we have used the fact that $H = T' T$, and suppose we choose $\delta_v = -K_p \delta_q$, with $K_p = (R' R)^{-1}$, then

$$\frac{d}{dt} \left(\frac{1}{2} \delta_z' \delta_z \right) = \delta_q (T' \dot{T} - C - K_D) \delta_q$$

Now recall that for the standard choice of Christoffel symbols, $\dot{H} - 2C$ is skew-symmetric, and $\dot{H} = T' \dot{T} + \dot{T}' T$, so we have

$$\frac{d}{dt} \left(\frac{1}{2} \delta_z' \delta_z \right) = -\delta_q K_D \delta_q$$

■

If the damping term K_D is artificial, then the construction of the differential control δ_v corresponds to a control signal

$$u = -K_d \dot{q} - \int_{\gamma} K_p \delta_q - G(q)$$

i.e. a “path integral” PD controller with gravity compensation, or other force fields. If the configuration space is \mathbb{R}^n then the integral is path independent and can be replaced by $K_p(q^* - q)$, giving a traditional PD controller, which can be thought of as a virtual spring and damper force field. If this state space is a nonlinear manifold, then the term $q^* - q$ may not have any meaning, a stabilizing control can still be computed via path integrals over the manifold.

Adjusting K_P and K_D shapes ellipsoidal forward-invariant regions of the differential dynamics, as well as their exponential rate of contraction. This gives a simple procedure to generate PD-like controls for mechanical systems on manifolds, and furthermore a way of shaping the closed-loop transient behaviour such as overshoot and settling time.

IV. SMALL GAIN THEOREMS AND DIFFERENTIAL L^2 -GAIN

The small-gain theorem is one of the cornerstones of rigorous analysis of feedback systems. It is the central idea of many important results in input-output theory, nonlinear stability, and H^∞ control.

Nevertheless, for general nonlinear systems there are reasons to prefer a stronger result than the standard small-gain result. In the influential paper [35], it was remarked by Zames that stability of a nonlinear system should be understood in terms of both boundedness of solutions and continuity of behaviour with respect to inputs. That is, small changes in the input to the system should result in small changes to the output. In their classic text on feedback systems [36], Desoer and Vidyasagar show that while the standard small-gain theorem only implies boundedness of solutions, an *incremental* small gain theorem implies that solutions are bounded, unique, and continuous with respect to inputs.

Oddly enough, for a long time the issue of continuity seems to have been almost forgotten in systems theory, with

boundedness of solutions and stability of certain special (e.g. equilibrium) solutions taking priority over continuity of solutions. In [10] there was a brief discussion of incremental integral quadratic constraints and their use in proving stability of periodic solutions. However, in [37] it was shown that certain “classic” multipliers used in IQC analysis, including the Popov and Zames-Falb multipliers, are not directly applicable for incremental stability analysis, since the “positivity preserving” property of these multipliers fails to be true.

In [38], Lyapunov-like conditions were derived for incremental input-to-state stability, giving a state-space property closely related to the input-output property of incremental small gain. Connections to small-gain theorems for input-to-state stable operators were studied in [39].

In this section we discuss the use of contraction theory and differential dissipativity [21] to analyse continuity of solutions.

Consider $y = Gw$ as a nonlinear operator from one normed function space to another. A common choice is $G : L^2[0, T] \rightarrow L^2[0, T]$ where T may be finite or infinite. Such a system is said to satisfy the *incremental small gain property* with gain bound α if for any pair of solutions $w_1 \rightarrow y_1$ and $w_2 \rightarrow y_2$, there exists a β such that

$$\|y_1 - y_2\| \leq \alpha \|w_1 - w_2\| + \beta.$$

The incremental small-gain theorem says that an interconnection of two such systems is stable if the product of their gain bounds is less than one. Specific versions of the theorem correspond to different choices of norm and, possibly, consideration of initial conditions.

In this paper, we consider the special case of state-space systems and the input spaces $L^2[0, T]$ and $L^2[0, \infty)$ and for simplicity we treat the “zero initial conditions” case, where $\beta = 0$. A system is said to satisfy an *incremental L^2 gain bound* of α if the following condition holds:

$$\int_0^T |y_1 - y_2|^2 dt \leq \alpha^2 \int_0^T |w_1 - w_2|^2 dt \quad (15)$$

for all $T > 0$ and all pairs of solutions $w_1 \rightarrow y_1$ and $w_2 \rightarrow y_2$ with $w_1, w_2 \in L^2[0, T]$.

For a given nonlinear operator G , if for each input $w \in L^2[0, T]$ there exists a linear operator DG_w satisfying

$$\lim_{\theta \rightarrow 0} \frac{G(w + \theta \delta_w) - Gw_1 - \theta DG_w \delta_w}{\theta} = 0$$

for all “differential inputs” $\delta_w \in L^2[0, T]$, then the operator is said to be Frechet differentiable and DG_w is its Frechet derivative at w . It follows from standard linearization analysis that for a smooth nonlinear system, taking as input an initial condition $x(0)$ and a driving input $w \in L^2[0, T]$, the Frechet differential is given by the differential dynamics - i.e. a time-varying linear system.

With this in mind, we say that a system satisfies a *differential L^2 gain bound* of α if

$$\int_0^T |\delta_y|^2 dt \leq \alpha^2 \int_0^T |\delta_w|^2 dt \quad (16)$$

for all T and all solutions of $\delta_y = DG_w \delta_w$ with $w, \delta_w \in L^2[0, T]$.

Theorem 5: For a Frechet differentiable nonlinear dynamical system, uniform incremental and differential L^2 gain – as defined in and (15) and (16) – are equivalent.

Proof: Suppose a system has uniform incremental L^2 gain α , i.e. for any pair of solutions w_1, y_1 and w_2, y_2 the relation holds

$$\int_0^T |y_1 - y_2|^2 dt \leq \alpha^2 \int_0^T |w_1 - w_2|^2 dt$$

Take the limit of a sequence of w_2 satisfying $\int_0^T |w_1 - w_2|^2 dt \rightarrow 0$ and, by definition of the Frechet derivative, the differential L^2 gain is bounded by α .

Conversely, suppose the system has uniform differential L^2 gain α . Choose a particular parametrization over θ of solutions to the system driven inputs $w_\theta = w_1 + \theta(w_2 - w_1)$ and the corresponding outputs y_θ as functions of time. Note that, by construction of the particular path as a straight line from w_1 to w_2 , $\delta_w(t) = \frac{\partial w_\theta}{\partial \theta}(t) = w(t)$ for all t . Now apply Lemma 1 to obtain

$$|y_1(t) - y_2(t)|^2 \leq \int_0^1 \left| \frac{\partial}{\partial \theta} y_\theta(t) \right|^2 d\theta$$

for each $t \in [0, T]$. The result then follows by integrating the above inequality. ■

We now specialise to the study of state-space models

$$\dot{x}(t) = f(x(t), t) + B_w(t)w(t) \quad (17)$$

with differential dynamics:

$$\delta_x(t) = A(x, t)\delta_x(t) + B_w(t)\delta_w(t) \quad (18)$$

where $A(x, t) = \frac{\partial}{\partial x} f(x, t)$ is the Jacobian matrix.

The following theorem can be considered a differential form of the “sufficiency” part of the bounded real lemma for linear systems [40].

Theorem 6: Suppose there exists a uniformly positive-definite metric $M(x)$ such that

$$\dot{M} + A'M + MA + \frac{1}{\alpha^2} M'B_w B_w' M + C'C \leq 0$$

Then the system is differentially L^2 bounded by α and, therefore, incrementally L^2 bounded by α .

Proof: By the Schur complement, this is equivalent to the matrix inequality

$$\begin{bmatrix} \dot{M} + A'M + MA + C'C & M'B_w \\ B_w' M & -\alpha^2 I \end{bmatrix} \leq 0$$

And since

$$\begin{aligned} \frac{d}{dt} (\delta_x' M \delta_x) &= \delta_x' (\dot{M} + A'M + MA) \delta_x + 2\delta_x' M B_w \delta_w \\ &\leq \alpha^2 |\delta_w|^2 - |\delta_y|^2. \end{aligned} \quad (19)$$

Integrating both sides gives

$$\begin{aligned} \delta_x(T)' M(x(T)) \delta_x(T) - \delta_x(0)' M(x(0)) \delta_x(0) \\ \leq \int_0^T \alpha^2 |\delta_w|^2 - |\delta_y|^2 dt. \end{aligned}$$

Now, since $M(x) > 0$ for all t we have

$$\int_0^T \alpha^2 |\delta_w|^2 - |\delta_y|^2 \geq -\kappa(x(0), \delta(0))$$

with $\kappa(x(0) = \delta_x(0)' M(x(0)) \delta_x(0)$. ■

Remark 2: While this constraint does not imply that the system is strictly contracting, it does imply that it is “output contracting”, i.e. any two solutions with the same input w and different initial conditions have the property that $y_1 \rightarrow y_2$.

By taking the transformation $W = M^{-1}$ and a Schur complement of the resulting Riccati differential equation we obtain a convex optimization procedure. That is, a bound on the systems (1) differential L^2 gain can be found by maximizing $\frac{1}{\alpha^2}$ subject to the following linear inequality:

$$\begin{bmatrix} (\dot{W} - W A' - A W - \frac{1}{\alpha^2} B_w B_w') & W C' \\ C W & I \end{bmatrix} \geq 0, \quad (20)$$

which is a convex function of the unknowns $W(x)$ and $\frac{1}{\alpha^2}$.

In the case of linear systems, this reduces to necessary and sufficient conditions for the H^∞ norm of a system to be less than γ .

V. ROBUST CONTROL DESIGN USING DIFFERENTIAL H^∞

In this section we consider the problem of designing a feedback control which stabilizes particular solutions of an uncertain system of the form:

$$\dot{x} = f(x, t) + B(t)u + B_w(t)w, \quad y = C(t)x(t). \quad (21)$$

The differential dynamics of the system are

$$\dot{\delta}_x = A \delta_x + B \delta_u + B_w \delta_w. \quad (22)$$

The uncertain feedback component is assumed to be Frechet differentiable, and satisfy the following differential IQC for all $T > 0$:

$$\int_0^T |\delta_w|^2 dt \leq \alpha_w^2 \int_0^T |\delta_y|^2 dt \quad (23)$$

where $\delta_y = C(t)\delta_x$. We call such a system *universally robustly stabilizable* if every feasible solution x^*, u^* of the system can be stabilized in the sense that the output error $|y - y^*|$ is in L^2 and the internal state remains bounded.

This class of uncertainties contains norm-bounded uncertainties common in H^∞ control, as well as nonlinear, time-varying dynamic uncertainties. This can be considered as a “differential” version of the type of uncertainty considered in [40].

Theorem 7: Consider the feedback interconnection of (21) and an uncertain system satisfying the differential IQC (23). Suppose there exists a solution to the pointwise LMI

$$\begin{bmatrix} (\dot{W} - W A' - A W + B B' - \rho B_w B_w') & W C' \\ C W & I \end{bmatrix} \geq 0, \quad (24)$$

with $\rho > \alpha_w^2$, then this system is universally robustly stabilizable.

Again, note that this condition is convex in the unknowns W and ρ . When restricted to linear time-invariant systems,

this is closely related to the conditions for H^∞ stabilization by state feedback given in [41].

Proof: Consider the family of solutions parameterized by s , where $\gamma(s) : [0, 1] \rightarrow \mathbb{R}^n$ is the minimal geodesic.

As previously, set $M = W^{-1}$ and apply Schur complement, then (24) is equivalent to the statement

$$\begin{bmatrix} \dot{P} + A'P + PA - PBB'P' + C'C & PB_w \\ B_w'P & -\frac{1}{\rho}I \end{bmatrix} \leq 0 \quad (25)$$

For the differential dynamics (22), set $M = W^{-1}$ and apply the differential feedback $\delta_u = -\frac{1}{2}B'M\delta_x$, then

$$\begin{aligned} \frac{d}{dt} \delta_x' M \delta_x &= \delta_x' \dot{M} \delta_x + 2\delta_x' M (A \delta_x + B \delta_u + B_w \delta_w) \\ &= \delta_x' (\dot{M} + A'M + MA - MBB'M) \delta_x \\ &\quad + 2\delta_x' M B_w \delta_w \\ &\leq \frac{1}{\rho} |\delta_w|^2 - |\delta_y|^2, \end{aligned} \quad (26)$$

where the last inequality comes from (25). Now,

$$\begin{aligned} d(T) - d(0) &= \int_0^T \dot{d}(x, x^*) dt \leq \int_0^T \int_0^1 \frac{d}{dt} (\delta' M \delta) ds dt \\ &\leq \int_0^1 \int_0^T -|\delta_y|^2 + \frac{1}{\rho} |\delta_w|^2 dt ds \\ &= \int_0^1 \frac{1}{\rho} \int_0^T |\delta_w|^2 - \alpha_w^2 |\delta_y|^2 - \beta |\delta_y|^2 dt ds \end{aligned} \quad (27)$$

where $\beta := \rho - \alpha_w^2 > 0$ by assumption. Furthermore, by the IQC assumption on the class of uncertainty inputs,

$$\int_0^T |\delta_w|^2 - \alpha_w^2 |\delta_y|^2 dt \leq 0$$

so the for each T , the result of the “inner” integration in (27) is ≤ 0 , so we have

$$d(T) + \beta \int_0^T \int_0^1 |\delta_y|^2 ds dt \leq d(0).$$

Now by Lemma 1, this then implies that

$$d(T) + \beta \int_0^T |y(t) - y^*(t)|^2 dt \leq d(0)$$

for all $T > 0$. Hence the feedback system is output L^2 bounded, and has bounded state. ■

As with (7), the explicit control signal is constructed by $u(t) = u^*(t) - \int_\gamma \frac{1}{2} B' M \delta_x$ where $\delta_x = \frac{\partial \gamma}{\partial s}$.

Remark 3: If the differential control $\delta_u = -\frac{1}{2}B'M\delta_x$ is integrable, then the resulting closed-loop system, considered as a nonlinear operator $w \rightarrow y$, has a differentially L^2 gain $\frac{1}{\sqrt{\rho}}$, as can be seen by comparing (26) with (19). Thus the condition $\rho > \alpha_w^2$ is precisely a “loop gain < 1 ” condition on the feedback interconnection. The main result of this theorem is that even if the differential control gain is *not* integrable, a robustly stabilizing controller can be constructed.

We should note that this is just one example among many similar constructions, in particular one related to input/output

L^2 gain. By using different supply rates one can additionally guarantee, e.g., exponential convergence of the. Another important example would come from a combination of the derivation given above with the results of [4] on differential guaranteed cost control, and the S-Procedure methods in [40], to give a guaranteed linear-quadratic cost controller for a class of uncertain systems described by differential IQCs.

It is also straightforward to extend the analysis to uncertainties with some “unknown initial conditions” by use of the differential IQC

$$\int_0^T |\delta_w|^2 dt \leq \alpha_w^2 \int_0^T |\delta_y|^2 dt + \kappa$$

where κ quantifies the effect of this uncertainty.

VI. OBSERVER DUALITY

It is well-known that the problems of control design and observer design for linear systems have a very attractive “duality” (see, e.g., [33]). In this section, we show that control contraction metrics admit a related duality with nonlinear observer designs using Riemannian metrics, as appearing for example in [18] and [42].

Now we consider the problem of state observers, for the non-autonomous system

$$\dot{x}(t) = f(x(t), t), \quad y = C(t)x(t) \quad (28)$$

with differential dynamics

$$\dot{\delta}_x = A(x, t)\delta_x, \quad \delta_y = C(t)\delta_x(t)$$

We will call a nonlinear system *universally detectable* if the following condition holds: any two solutions $x_1(t), x_2(t)$ that induce identical outputs $y_1(t) = y_2(t) \forall t$ have the property that $x_1(t) \rightarrow x_2(t)$ as $t \rightarrow \infty$. I.e. indistinguishable states are convergent.

If we take as a given the linear theory of controller/observer duality, and simply replace $A(x, t)$ with $A(x, t)'$ and $B(t)$ with $C(t)'$ in (10) we obtain the condition

$$\dot{W} + A'W + WA - \rho C'C < 0, \quad (29)$$

where W is uniformly positive-definite.

We note that a similar condition has appeared before in the literature giving necessary conditions for existence of a particular class of observer [42]. By Finsler’s theorem, (29) is equivalent to the statement that

$$C(t)\delta_x(t) = 0 \implies \frac{d}{dt}[\delta'_x W(x, t)\delta_x] < 0.$$

That is, in directions orthogonal (with respect to W) to the subspace spanned by the columns of $C(t)$, the system is contracting. Note that in this conditions $W(x, t)$ is the contraction metric, whereas in (10) $M(x, t) = W(x, t)^{-1}$ was the contraction metric.

A. Construction of an Observer

Suppose condition (29) is satisfied, and choose an initial state estimate $\hat{x}(0)$. For a given time t , define the set $\mathfrak{X}_y(t) := \{x : C(t)x = y(t)\}$ as the set of states consistent with the current measurement. Now let $\gamma(s, t)$ be the shortest path, with respect to the metric $\delta'W\delta$, between $\hat{x}(t)$ and the set $\mathfrak{X}_y(t)$.

Then construct an observer with the following dynamics:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), t) + \int_0^T K(\gamma(s))\delta_y(s)ds \quad (30)$$

where

$$K(x) = \frac{1}{2}\rho(x, t)W(x, t)^{-1}C(t)'$$

Note that if W and ρ are independent of x , this reduces to a standard Luenberger-type observer

$$\dot{\hat{x}}(t) = f(\hat{x}(t), t) + K(t)(y(t) - C(t)\hat{x}(t)).$$

Theorem 8: The state $\hat{x}(t)$ of the observer given in (30) converges to a solution $x(t)$ of (28) satisfying $Cx(t) = y(t)$.

Proof: We give a sketch here due to space restrictions. The differential dynamics along the curve $\gamma(s)$ are given by

$$\dot{\delta}_x = [A(\gamma, t) - \frac{1}{2}\rho(x, t)W(x, t)^{-1}C(t)'C(t)]\delta_x$$

which gives

$$\frac{d}{dt}\delta'_x W \delta_x = \delta'_x (\dot{W} + A'W + WA - \rho C'C)\delta_x < 0$$

where the inequality comes from condition (30). It follows that the length of the path $\gamma(s)$ is decreasing, and therefore by standard Lyapunov arguments and the uniform positive-definiteness of W , and Lemma 1, the system $\hat{x}(t)$ converges to the set $\mathfrak{X}_y(t)$, so $C\hat{x} \rightarrow y$. Moreover, the observer dynamics are smooth and satisfy $\dot{\hat{x}} = f(\hat{x}, t)$ if $\hat{x} \in \mathfrak{X}_y(t)$. ■

It is clear from the above that any differential gain $\alpha(x)K(x)$ with $\alpha(x) > 1$ is also convergent, and this corresponds to a form of “infinite gain margin” of the observer.

In [42] conditions of the form (30) were discussed, and it was shown that geodesic convexity is necessary for an observer to converge with respect to a particular metric. In relation to this we can give the following result:

Theorem 9: If the conditions of Theorem 8 hold and, additionally, the set $\mathfrak{X}_y(t)$ is geodesically convex with respect to the metric $W(x, t)$, then the observer constructed above converges, i.e. $\hat{x}(t) \rightarrow x(t)$. By construction, the system is universally detectable.

Proof: From above, the observer state converges to a state satisfying $\mathfrak{X}_y(t)$. Furthermore, by Finsler’s theorem, the condition (30) implies that all differentials lying entirely in $\mathfrak{X}_y(t)$ are contracting. But if this set is geodesically convex, then the shortest curve between any two points in $\mathfrak{X}_y(t)$ lies entirely in $\mathfrak{X}_y(t)$, and therefore so do its differentials. Hence all states in $\mathfrak{X}_y(t)$ converge, and so the observer state converges to the true system state. ■

This observer is closely related to the observer proposed in [42]. To make a direct comparison, the observer in [42] makes use of the same pointwise LMI condition, but gives a simpler observer construction not requiring computation of a geodesic or path integration. The price paid for the simpler construction is that their result is local convergence, whereas ours is global, and their result is restricted to systems with a scalar nonlinear output map, whereas ours allows multidimensional outputs.

Remark 4: The above result can be extended to systems with nonlinear output maps $y(t) = c(x(t), t)$ without modification, so long as c satisfies certain “regularity” conditions ensuring that the manifold $\mathfrak{X}_y(t)$ and the minimal path γ are well-defined. We omit the details, which are similar to those in [42].

Remark 5: As with the problem of control design, an exponentially convergent observer can be obtained by a stronger condition

$$\dot{W} + A'W + WA - \rho C'C < -2\lambda W. \quad (31)$$

Assuming additionally that W and C are uniformly bounded, the same proof procedure can be adapted to show that $y - C\hat{x}$ converges to zero exponentially with rate λ and, in the case where $\mathfrak{X}_y(t)$ is geodesically convex, that $x - \hat{x}$ converges to zero with exponential rate λ .

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